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A BEM BASED PATTERN SEARCH SOLUTION FOR A CLASS OF INVERSE ELASTOSTATIC PROBLEMS

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Abstract—A boundary element based solution, employing the Hooke-Jeeves pattern search method, is developed to determine internal cavity geometries in the inverse elastostatics problem. In contrast to existing methods which employ gradient based solutions, the developed method does not require gradient information, and consequently results in a simpler solution procedure. A boundary difference function (BDF) that provides a guide for the initial guess is introduced. The unknown cavity geometry is first assumed to be circular, and the algorithm is run until the general location of the cavity is determined. Subsequent refinement of the detected cavity geometry is affected using a periodic cubic spline discretization. Numerical results are presented to illustrate the effectiveness of the boundary difference function and to demonstrate the successful detection of the cavity. The effects of simulated experimental inputs with prescribed error are also presented. Copyright © 1996 Elsevier Science Ltd.

INTRODUCTION

In a forward problem the governing equation, the boundary conditions, the material properties and the system geometry are all specified. The purpose of the forward problem is to determine the field variable. In an inverse problem, a portion of the above is unspecified and determined with the aid of overspecified conditions. Typically applications of inverse problems include system parameter identification, e.g. Schnur *et al.* (1992) and Ikehata (1990). An extensive review of inverse problems and other applications can be found in the recent monograph by Bui (1994). A class of inverse problems concerned with determining unknown system geometry finds application in nondestructive evaluation. This class of problems is collectively referred to as the inverse geometric problem.

The purpose of the inverse geometric problem is to determine the hidden portion of the system geometry by using overspecified boundary conditions on the exposed portion. This problem has gained importance in thermal and solid mechanics applications for the nondestructive detection of subsurface cavities, e.g. Kassab *et al.* (1995). In thermal applications, the method requires overspecified boundary conditions at the surface, i.e., both temperature and flux must be given, e.g. Kassab and Pollard (1994). In elastostatics applications, the overspecified conditions are given in terms of surface displacements and tractions. Generally, surface tractions are known boundary conditions, while the surface displacements are experimentally determined, for example, by laser speckle photography in the paper by Kassab *et al.* (1994).

Given the governing field equations and the overspecified boundary conditions, the inverse geometric problem is solved iteratively. At each iteration a forward field problem employing the currently updated geometry is solved. The solution of the forward problem can be obtained using either the finite element method (FEM), Maniatty *et al.* (1989), or the boundary element method (BEM), Kassab *et al.* (1994) and Bezerra and Saigal (1993). However, BEM offers a distinct advantage, as only boundary discretization is required. This is particularly pronounced in our application where the boundary continuously evolves in the solution process. Typically, the iteration process updates the current cavity geometry

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by minimizing an objective function. This objective function measures the level of non-satisfaction of the overspecified boundary conditions. Currently, researchers rely on gradient based techniques because of their convergence characteristics. Examples of these gradient based algorithms are found in Tanaka *et al.* (1988) and Tanaka *et al.* (1991) for elastodynamics applications, and in Kassab *et al.* (1994), Bezerra and Saigal (1993) and Tanaka and Masuda (1986) for elastostatics applications. These gradient based methods require the evaluation of sensitivity coefficients which can be determined either numerically by using finite differences, see Kibsgaard (1992), or analytically by differentiating the BEM equations, see Mukherjee and Zhang (1994). The former can lead to poor resolution of the sensitivity coefficients, while the latter leads to complicated hypersingular integrals, see Saigal *et al.* (1989).

In this paper, a boundary element based algorithm, employing the Hooke-Jeeves pattern search method, is developed to solve the inverse elastostatics problem (IESP) and determine internal cavity geometries. This method does not require gradient information and, consequently, results in a simpler solution procedure. A boundary difference function (BDF) is also formulated to provide a guide for the initial guess. The unknown geometry is detected in a two stage process. First, the unknown cavity geometry is assumed to be circular, and the algorithm is run until the general location of the cavity is determined. Subsequent refinement of the detected cavity geometry is effected using a periodic cubic spline discretization. Numerical results are presented to illustrate the effectiveness of the BDF as a guide in the placement of the initial guess. Results demonstrate the successful detection of an irregular subsurface cavity, including cases simulating the effects of error in experimental measurements of input data.

BOUNDARY ELEMENT METHOD

The boundary element method (BEM) for two-dimensional elastostatics problems is well known and further details can be found in Brebbia *et al.* (1984) and Banerjee (1994). In this paper, the direct BEM formulation is used and is briefly reviewed below. The governing equation (Navier equation) for an elastostatics problem in the absence of body forces is

$$\nabla^2\mathbf{u}+\frac{1}{1-2\nu}\nabla(\nabla\cdot\mathbf{u}) = 0 \tag{1}$$

where $\mathbf{u}$ are the displacements and ν is the Poisson ratio. Introducing the fundamental solution to the Navier equation and using a reciprocity relation, a boundary integral equation relating boundary displacements $\mathbf{u}$ and the boundary tractions $\mathbf{t}$ can be derived as

$$\mathbf{cu}+\oint_\Gamma \mathbf{t^*u}\,\mathrm{d}\Gamma = \oint_\Gamma \mathbf{u^*t}\,\mathrm{d}\Gamma \tag{2}$$

where $\mathbf{u^*}$ and $\mathbf{t^*}$ are the fundamental displacements and tractions respectively, and $\mathbf{c}$ is a constant matrix, see Brebbia *et al.* (1984). Employing standard boundary element procedures, the above equation is written in discrete form as

$$[\mathbf{H}]\{\mathbf{u}\} = [\mathbf{G}]\{\mathbf{t}\} \tag{3}$$

where $[\mathbf{H}]$ and $[\mathbf{G}]$ are influence coefficient matrices, $\{\mathbf{u}\}$ is the boundary displacement vector, and $\{\mathbf{t}\}$ is the boundary traction vector. Applying boundary conditions and separating the knowns and unknowns, the above equation can be arranged in standard form as $[\mathbf{A}]\{\mathbf{x}\} = \{\mathbf{b}\}$. This system of equations is solved using Gauss elimination.

ITERATIVE SOLUTION OF THE IESP

The iterative solution of the inverse elastostatics problem (IESP) by a pattern search based algorithm is now developed. An initial guess is first made for the interior cavity, and the forward BEM solution of the elastostatics problem is carried out. Obviously, unless the initial guess matches the actual cavity geometry, the computed surface displacements will differ significantly from the actual values, which are hereafter referred to as the reference values. Consequently, an objective function, F, that quantifies this difference is defined as

$$F = \sum_{i=1}^{m} \left[\Delta u_i + \frac{1}{2\mu} \Delta t_i \right] \tag{4}$$

where

$$\Delta u_i = \sqrt{[(u_{i_{ref}} - u_i)^2 + (v_{i_{ref}} - v_i)^2]} \tag{5}$$

$$\Delta t_i = \sqrt{[(t_{x_{ref_i}} - t_{x_i})^2 + (t_{y_{ref_i}} - t_{y_i})^2]}. \tag{6}$$

Here, Δu_i is the length of the displacement difference vector, Δt_i is the length of the traction difference vector, i refers to the node number, and *ref* refers to the reference boundary quantity. Notice that the above objective function contains both traction and displacement differences. This is due to the fact that the tractions appear as unknown at those points which are constrained to prevent rigid body motion. The scaling factor $(1/2\mu)$, where μ is the shear modulus, is used to adjust the magnitudes of the tractions and displacements to be of comparable values.

Closer examination of the objective function reveals that the coefficients of the parametric representation of the current cavity implicitly appear as dependent variables. As such, minimizing the objective function will automatically update the current cavity geometry in order to satisfy the imposed boundary conditions.

The Hooke-Jeeves pattern search method, which is discussed in Reklaitis *et al.* (1983), is used to minimize the objective function, F, in the iterative process which updates the cavity geometry, and we now briefly describe this method. The Hooke-Jeeves method requires an initial starting point for the dependent variables and an initial step size. The start point is initially set as the center point for the pattern search, and the objective function is evaluated there. A step is taken from the center point of the pattern search along the positive direction for each dependent variable, and this process is repeated in the negative direction. A cycle is completed when the algorithm has taken a step in the positive and negative directions for all design variables. During a cycle, if any step results in an improved value for the objective function as compared to that of the center point, then the center point is shifted there. The remaining cycle uses this latest center point. Subsequent to the completion of a cycle, a step line search is undertaken along a direction defined by the current start point and the best point to date. The best point provided by this step line search is set as the new start point. The pattern search process is then repeated. If a cycle produces no improvement, the step size is reduced in half, and the process is repeated until the step size reaches a stopping tolerance. This process is illustrated for two dependent variables in Fig. 1.

The parameterization of the cavity is accomplished differently in each of the two stages of detection. In the first stage, the cavity is effectively parameterized as a circle. Consequently, the x and y coordinates of the center and the radius are the dependent variables. Upon successful location of the position and general size of the cavity, a refined second stage parameterization is implemented. Here, the cavity is defined by a local origin anchoring radial lines upon which a periodic cubic spline is used to describe the cavity perimeter, see Fig. 2. To determine convergence, a distance is defined as the magnitude of the difference vector of the design variables for the present step and the immediate prior step. The search process is completed when a preset tolerance of $e = 0.0001$ for both the

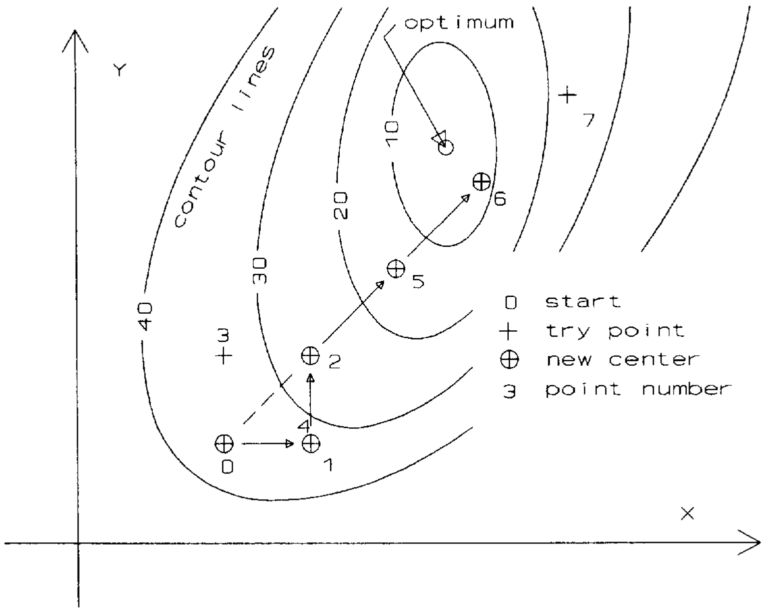

Fig. 1. Hooke-Jeeves pattern search.

step size and distance is reached, or when a preset maximum number of BEM calls is exceeded. In this study we allow for a maximum of 2000 BEM calls.

The actual implementation is a cascade search with cavity size and center as the first stage variables, and the second stage with the periodic cubic spline knots as variables. The first stage is repeated until the norm measuring the distance between the vector of dependent variables from one iteration to another falls below a specified tolerance of $e = 0.01$. The second stage is then run for a single step. The first stage is subsequently implemented to

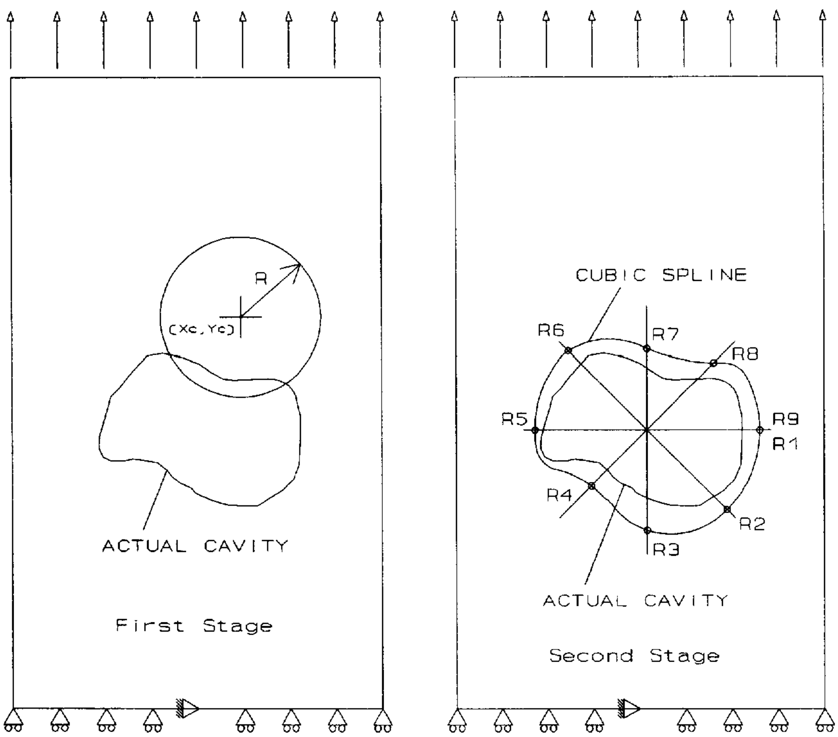

Fig. 2. Cascading solution stages.

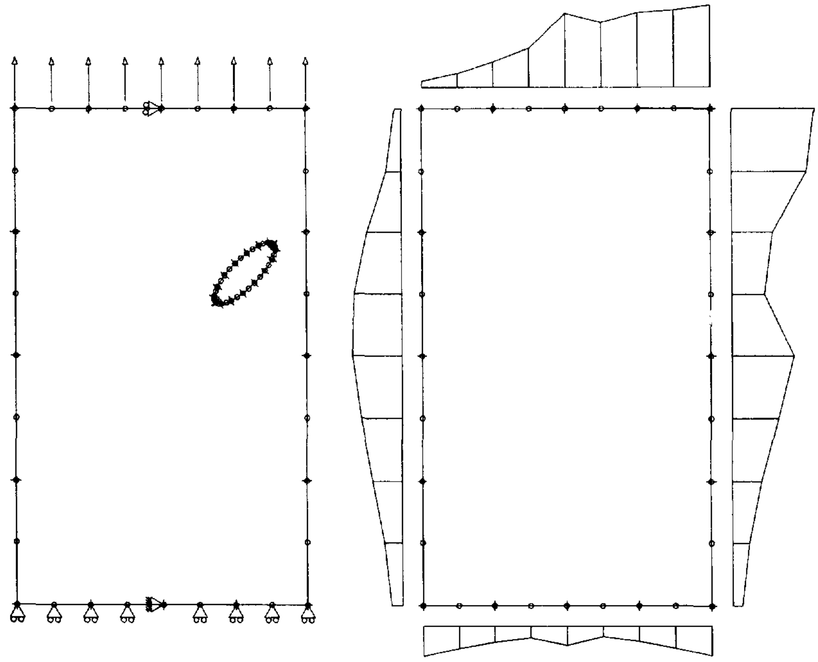

Fig. 3. (a) Off-set elliptic cavity and loading. (b) BDF plot.

center the new cavity shape. The process continues alternating between the two stages, until final convergence is reached.

THE BOUNDARY DIFFERENCE FUNCTION (BDF)

A boundary difference function (BDF) is introduced to provide a good initial guess. The BDF is defined with the aid of the objective function, eqn (4). In particular, a forward BEM problem is solved assuming no cavity. The computed displacements and tractions for this case are then input in the objective function. At this point, the objective function measures the difference between the actual boundary conditions and those computed under the no cavity assumption. We refer to this function as the BDF. Clearly, the BDF indicates whether a cavity exists at all. Specifically, the absence of included cavities is indicated by a BDF value of zero (or nearly zero due to experimental error) along the boundary. Further, for the case of an included cavity, the plot of the BDF along the exposed boundary proves to be quite informative. It provides guidance as to the general location of included cavities and aids in the placement of the initial starting point.

A typical BDF is illustrated in Fig. 3 for the case of an offset tilted elliptic cavity included within a plate under uniaxial tension. The presence of a cavity is clearly illustrated by the BDF. Further, the location of the cavity in the upper right hand corner is indicated by the magnitude and abrupt changes in the BDF curve in that vicinity. This suggests that the starting point should be located somewhere in the upper right hand quadrant. This illustrates the heuristic approach to using the BDF in placing the initial guess. Attention is now given to numerical simulation for the verification of the above developed algorithm.

NUMERICAL EXAMPLES

For the first example we consider the detection of the offset elliptic cavity shown in Fig. 3. The start, some intermediate steps, and comparison of actual and detected cavity

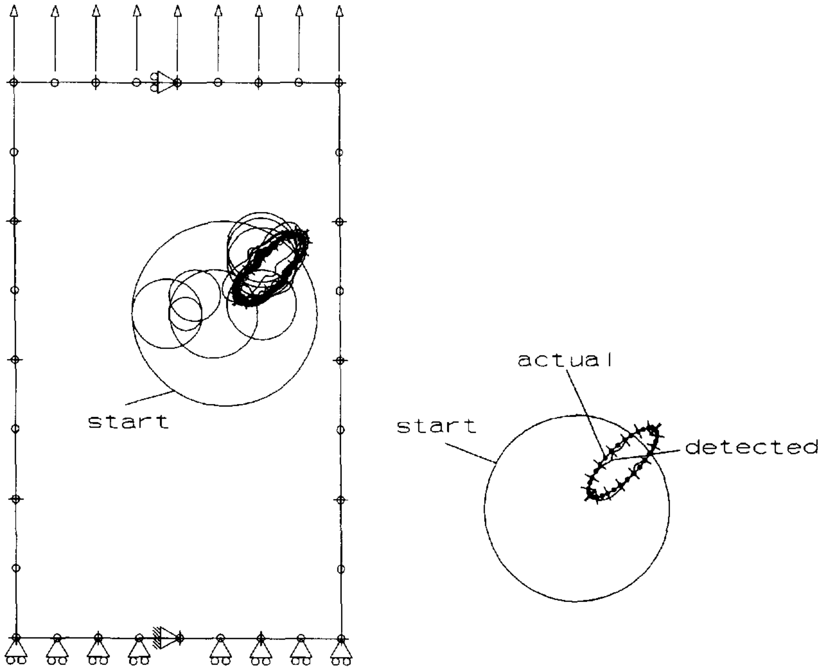

Fig. 4. Iterative solution for the offset elliptical cavity.

shapes are shown in Fig. 4. A circle of radius $R = 1.0$ located at (0.5, 0.5) was used as a starting point. The cavity is closely detected in 53 steps.

As an example of a poorly placed starting guess, a circle of radius $R = 1.0$ located at $(-0.5, -1.75)$ was used as shown in Fig. 5. This starting guess is placed in contradiction to the indications of the BDF as previously discussed. Selected resulting steps are shown in Fig. 5. The search algorithm stops due to a failure to improve on previous steps. Clearly, the ending geometry in not close to the actual cavity.

In the second example, we consider detecting the irregularly-shaped cavity enclosed within a plate under biaxial tension as illustrated in Fig. 6. The BDF for this case indicates a cavity somewhere near the center. Consequently, an initial start using a circular cavity of radius $R = 1.0$ located at (0.0, 0.0) is used. The start, some intermediate steps, and comparison of actual and detected cavities are shown in Fig. 7. The algorithm converged within tolerance in 152 steps. There is good agreement between the detected and actual cavities.

An immediate concern arises as to the robustness of the algorithm to input error. As such, we now consider an example simulating input error in displacements within the range expected from laboratory measurements. A $\pm 5\%$ bias error is added to the reference displacements, and these are used as inputs. Here, an initial start using a circular cavity of radius $R = 1.0$ located at (0.0, 0.0) is consistently used for all cases. The resulting detected cavities are shown in Fig. 8. Clearly, the algorithm is shown to be robust to input error, and the numerical validation of the algorithm is thus successful.

CONCLUSIONS

A boundary element based solution, using the Hooke-Jeeves pattern search method, is developed to determine internal cavity geometries in the inverse elastostatics problem (IESP). A boundary difference function (BDF) is introduced to provide a guide for the initial guess. The numerical results show the effectiveness of the BDF. The algorithm successfully detected an offset elliptic cavity and an irregular cavity using exact inputs. The robustness of the algorithm to input error is also demonstrated by successfully detecting a cavity with simulated input error in displacements within the range expected from laboratory measurements.

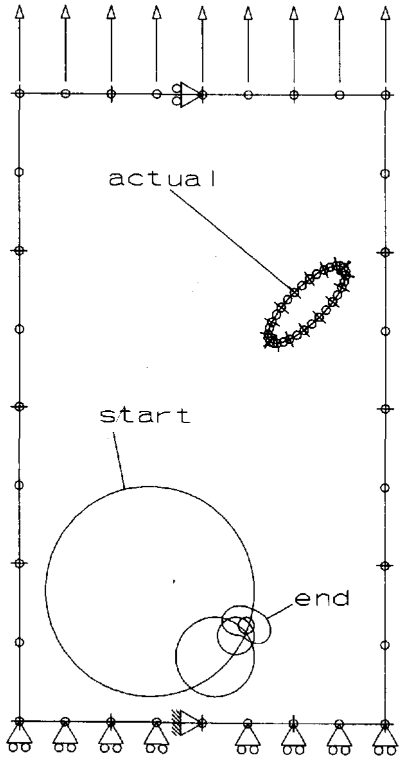

Fig. 5. Example of a poorly-placed starting guess.

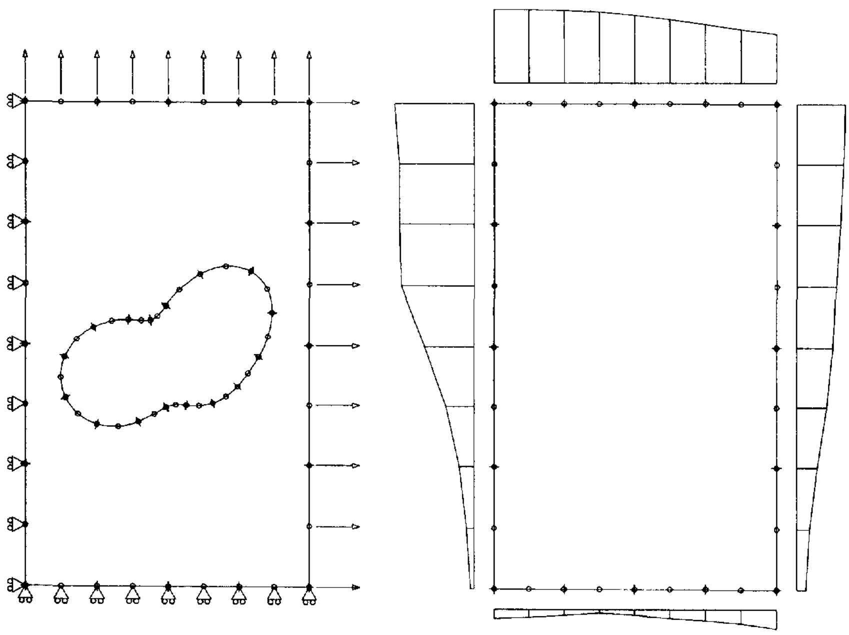

Fig. 6. (a) The irregularly-shaped cavity example. (b) BDF plot.

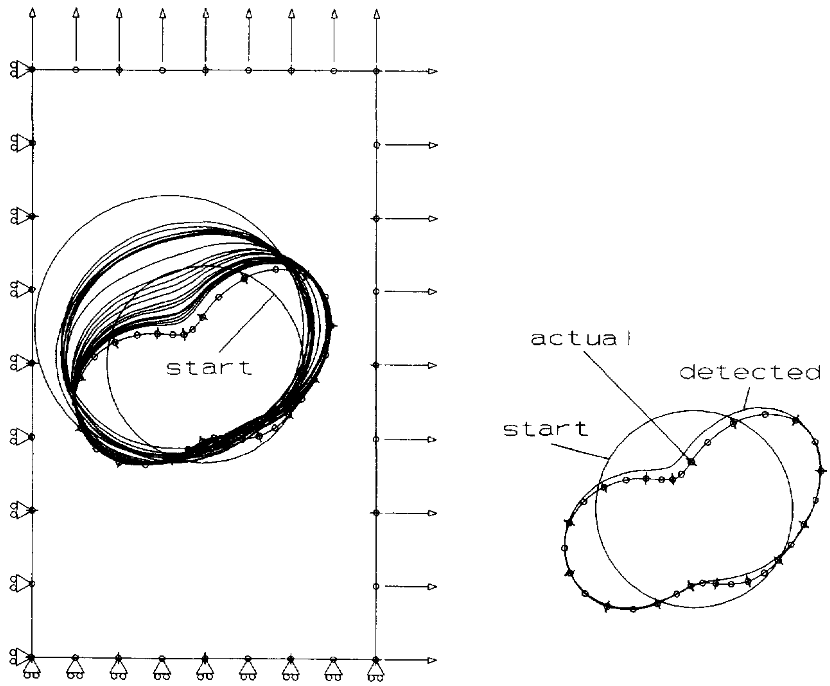

Fig. 7. Iterative solution for irregularly-shaped cavity.

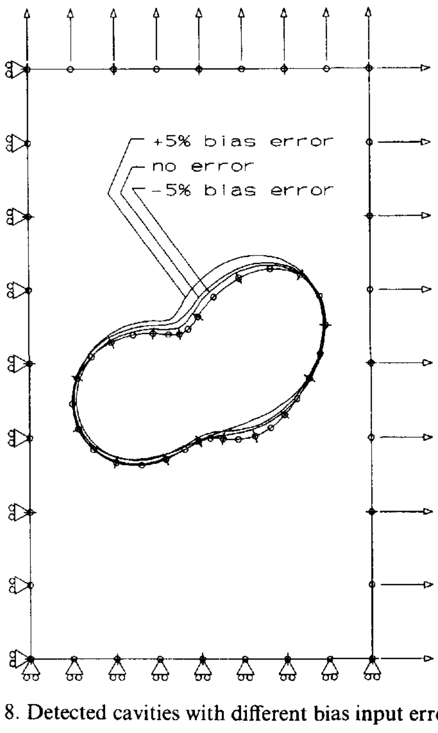

Fig. 8. Detected cavities with different bias input errors.

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